Open-system theory of photodetection: dynamics of field and atomic moments

1978 J. Phys. A: Math. Gen. 11 1427

(http://iopscience.iop.org/0305-4470/11/7/029)

J. Phys. A: Math. Gen., Vol. 11, No. 7, 1978. Printed in Great Britain

Open-system theory of photodetection: dynamics of field and atomic moments

Annabella Selloni†, P Schwendimann‡, A Quattropani† and H P Baltes§

† Laboratoire de Physique Théorique, Ecole Polytechnique Fédérale Lausanne, CH-1006 Lausanne, Switzerland

‡ Institut für Theoretische Physik, Universität Bern, CH-3000 Bern, Switzerland

§ Zentrale Forschung und Entwicklung, Landis & Gyr Zug AG, CH-6301 Zug, Switzerland

Received 25 November 1977, in final form 23 February 1978

Abstract. A theory of photodetection is presented which takes fully into account the relaxation of the photosensitive atoms representing the detector. Atoms, electromagnetic field, and atom–field interaction are described in terms of the Dicke Hamiltonian with additional coupling of the atoms to a reservoir. The time evolution of the joint detector and field states is derived using master equation techniques. Simple relations between the dynamics of field and atomic moments are obtained in the case of zero detector temperature. The attenuation of the field moments as well as the photocounting probability for any initial field state are found to involve not only the atom–field, but also the atom–reservoir coupling parameters.

1. Introduction

In this article we reinvestigate the problem of retrieving statistical properties of electromagnetic fields from photodetector signals. Previous detector model Hamiltonians include the detector atoms, the field, and the atom–field interaction. Two different approaches can be distinguished in the literature: semi-classical perturbation theory (Mandel 1963, Mandel *et al* 1964, Wolf and Mehta 1964) and full quantum perturbation theory (Kelly and Kleiner 1964, Glauber 1965, 1969, Lehmberg 1968, Mollow 1968, Scully and Lamb 1969, Jaiswal and Agarwal 1969, Mandel and Meltzer 1969, Rocca 1973, Arnedo and Rocca 1974, and Rousseau 1977). Reviews accounting for the corresponding states of the art (Mandel and Wolf 1965, Mehta 1970, Arecchi and Degiorgio 1972) and for the link with decision theory (Helstrom 1972) are available. The interest in applications of photon correlation such as Doppler velocimetry (Cummins and Pike 1974, Watrasiewicz and Rudd 1975) and scattering by random phase screens (Cummins and Pike 1974, Zardecki and Delisle 1977, Mandel and Wolf 1977) is still growing.

Conventional perturbation theory (Kelly and Kleiner 1964, Glauber 1965, Rocca 1973, Arnedo and Rocca 1974) excludes the effects of virtual transitions and requires a reservoir-like behaviour of the field, since the attenuation of the field due to the interaction with the detector is not adequately taken into account. Other procedures

0305-4770/78/0007-1427$02.00

such as the Heisenberg equation of motion (Mollow 1968, Glauber 1969) and master-equation-type methods (Scully and Lamb 1969) fully account for the field attenuation, but require the atomic system to act as a reservoir. Thus the atoms have to play the role of the photosensitive system and, simultaneously, that of the reservoir. This leads to severe limitations being imposed on the atomic spectrum, the atom–field coupling strength, and the lifetime of excited states (Mollow 1968).

In the present paper we propose a detector scheme in which the atoms are not only coupled to the field as usual, but are in addition coupled to a reservoir. The reservoir provides the mechanism for the relaxation of excited atoms, i.e. the withdrawal of photoelectrons. Thus the coupling of the atoms to the reservoir prevents radiative decay during the measurement as well as back reaction effects like stimulated emission and saturation. In experimental situations the reservoir corresponds to the detector electronics and cryostat.

Our model calculation shows the following features. We consider a point-like model detector that consists of many identical two-level atoms. We describe the atoms, the field, and the atom–field interaction in terms of the Dicke (1954) Hamiltonian. We introduce additional terms in the Hamiltonian which allow for the coupling of the atoms to a reservoir with rapidly decaying internal correlation. We derive the dynamics of the joint states of the detector atoms and the field by using the reduced-density-matrix formalism for the successive elimination of reservoir and atomic, or field, variables. We assume that the atom–field coupling is strong compared with the atom–bath coupling and consider times that are large compared with the atomic relaxation time. Moreover, we establish the attenuation of the field moments as well as the relations between atomic and field moments of any order in the case of zero detector temperature. Temperature effects on the field dynamics are studied in a forthcoming paper. Finally, we find the photocounting rate to be formally analogous to the one derived by Mollow (1968) and Scully and Lamb (1969) and Rousseau (1977), with the conversion efficiency, however, depending not only on the atom–field, but also the atom–bath, coupling parameter. Our approach accounts adequately for both the field attenuation and the atomic relaxation.

In § 2 we present our detector model, and in § 3 we derive the equation of motion of the reduced density operator of the Dicke system by eliminating the bath variables. The objective of §4 is the field attenuation in the case of zero detector temperature. The connection between the dynamics of atomic and field moments is established in § 5. Finally, in § 6, we discuss the corresponding photocounting probability.

The generalisation of our approach to multi-mode fields and different atomic positions (extended detector) and level spacings is straightforward, but is not presented here.

2. Model Hamiltonian

We assume that the photodetector consists of N independent equivalent, but distinguishable, two-level atoms with level spacing ϵ. The atoms interact with a single-mode field of frequency ω in a cavity of volume V. Atoms, field, and interaction are described by the Dicke Hamiltonian

$$\begin{aligned} H_D &= H_F + H_A + \lambda V^{-1/2} H_{AF} \\ &= \omega a^+ a + \epsilon S_3 + \lambda V^{-1/2}(aS^+ + a^+ S^-), \end{aligned} \tag{1}$$

where $a^\dagger$, a denote the field operators and where

$$S_3 = \sum_{j=1} s_{3j} \qquad \text{and} \qquad S^\pm = \sum_{j=1}^{N} s_j^\pm$$

denote the collective, and s_{3j} and $s_j^\pm$ the individual, atomic variables. The coupling constant is $\lambda = \omega^{1/2} g_{12}$ with g_{12} denoting the dipole transition matrix element of the atoms ($\hbar \equiv 1$). In the present paper we assume that the atoms are initially in their ground state $|0\rangle_A \equiv |S = N/2, S_3 = -N/2\rangle$. An interaction of the atoms with a reservoir is required in order to provide a stationary distribution of the atoms in the initial state and a mechanism for withdrawing excited atoms and allowing the definition of a photoelectric current. We therefore introduce a bath described by the Hamiltonian H_B and its interaction with the atoms described by the Hamiltonian H_{AB}. The full detector model is thus given by

$$H = H_D + H_B + H_{AB}. \tag{2}$$

In order to prevent reservoir-induced interaction between the atoms we couple each atom to an individual bath described by the observables Γ_j^+, Γ_j^-. This is achieved by choosing the coupling Hamiltonian

$$H_{AB} = \sum_{j=1}^{N} \mu_j (\Gamma_j^- s_j^+ + \Gamma_j^+ s_j^-) \tag{3}$$

with μ_j denoting the coupling constants. We assume that the bath observables obey the relation

$$\exp(iH_B t)\Gamma^\pm \exp(-iH_B t) = \sum_{n,m} \Gamma_{nm}^\pm \exp[i(E_n - E_m)t] |n\rangle\langle m| \tag{4}$$

with diagonal matrix elements $\Gamma_{nn}^\pm = 0$, and the commutator or anticommutator relation $[\Gamma_i^+, \Gamma_j^-]_\pm = \delta_{ij}$. All other commutators or anticommutators are assumed to vanish. We do not have to specify whether the bath is made up of bosons or fermions.

The above Hamiltonian simplifies provided that the detector temperature is zero and the number $\langle N_2 \rangle$ of excited atoms is small, i.e. $\langle N_2 \rangle \ll N$. In this limit, the atomic system can be described as well by the oscillator approximation (Glauber 1965, Arecchi *et al* 1972, Selloni *et al* 1977).

3. Elimination of bath variables

Let us denote by ρ the density operator which corresponds to the total system described by the Hamiltonian (2) with (1) and (3). Applying the partial trace with respect to the bath variables as usual, we obtain the reduced density matrix $\rho_D \equiv \mathrm{Tr}_B \rho$ which describes the Dicke system whose atoms are affected by interaction with the reservoir. We study the time evolution of ρ_D using master-equation methods (Zwanzig 1961, Favre and Martin 1968, Haken 1970, Haake 1973). To this end, we assume that the bath and the Dicke system are uncorrelated at time $t = 0$ and that the initial bath equilibrium $\rho_B(0)$ is stationary with respect to the free bath motion (Zwanzig 1960, Favre and Martin 1968). We consider the equation of motion of ρ_D in the van Hove (weak-coupling) limit leading to Markoffian time evolution. We make use of the asymptotic form (Scharf 1970, Schwendimann 1972) of the eigenvalues $E_n(\lambda)$ of the Dicke Hamiltonian for large N, namely, $E_n(\lambda) \sim E_n(\lambda = 0) + n\lambda$. Moreover, we have

to require that any internal bath correlations (Louisell 1973) decay in a time that is small compared with the characteristic time involved in the dynamics of the Dicke system. Approximating the bath correlation function by exponential decay, e.g.

$$\langle \Gamma_j^+(t)\Gamma_j^-(0)\rangle_B \sim \exp(-t/\tau_j), \tag{5}$$

where the average is meant with respect to bath equilibrium distribution $\rho_B(0)$, we have to assume that the condition (Schwendimann 1972)

$$\tau_j \ll (V/N)\lambda^{-1} \tag{6}$$

is fulfilled. Now we are entitled to approximate $\exp(\mathscr{H}_D t)$ by $\exp(\mathscr{H}_A t)\exp(\mathscr{H}_F t)$ with $\mathscr{H}_D$, $\mathscr{H}_A$, and $\mathscr{H}_F$ denoting the Liouvillians corresponding to H_D, H_A, and H_F, respectively. We thus obtain the equation of motion of the reduced density operator ρ_D, namely

$$\dot{\rho}_D(t) = -i[H_D, \rho_D(t)] + \mathscr{D}_A \rho_D(t), \qquad \mathscr{D}_A \equiv \sum_{j=1}^{N} \mathscr{D}_j, \tag{7}$$

with Banach-space operators $\mathscr{D}_j$ defined by

$$\mathscr{D}_j \rho_D \equiv \tfrac{1}{2}\{\gamma_{1j}([s_j^-, s_j^+\rho_D] + [\rho_D s_j^-, s_j^+]) + \gamma_{2j}([s_j^+, s_j^-\rho_D] + [\rho_D s_j^+, s_j^-])\}. \tag{8}$$

Here γ_{1j} and γ_{2j} are inverse atomic relaxation times which depend on the atom–reservoir coupling strengths μ_j and on the reservoir correlation functions, namely

$$\gamma_{1j} \equiv \mu_j^2 \int_0^\infty \langle \Gamma_j^+(0)\Gamma_j^-(-t)\rangle_B \exp(-i\epsilon t)\, dt, \tag{9}$$

$$\gamma_{2j} \equiv \mu_j^2 \int_0^\infty \langle \Gamma_j^-(-t)\Gamma_j^+(0)\rangle_B \exp(i\epsilon t)\, dt, \tag{10}$$

where the index B stands for the free-bath equilibrium state $\rho_B(0)$. We observe that the non-Hermitian operators $\mathscr{D}_j$ have the properties

$$\mathscr{D}_j s_j^\pm = -\tfrac{1}{2}(\gamma_{1j} + \gamma_{2j}) s_j^\pm, \tag{11}$$

$$\mathscr{D}_j s_{3j} = -(\gamma_{1j} + \gamma_{2j})(s_{3j} - \sigma_{0j}/2) \tag{12}$$

where $\sigma_{0j} \equiv (\gamma_{1j} - \gamma_{2j})/(\gamma_{1j} + \gamma_{2j})$.

Since our model (1) requires identical atoms, we assume that the atomic relaxation times are identical, i.e. we are entitled to write

$$\gamma_1 \equiv \gamma_{1j}, \qquad \gamma_2 \equiv \gamma_{2j}, \qquad \sigma_0 \equiv \sigma_{0j}. \tag{13}$$

In the case of a thermal bath with $\rho_B(0) \propto \exp(-\beta H_B)$ we have $\gamma_1/\gamma_2 = \exp(-\beta\epsilon)$ and therefore

$$\sigma_0 = -\tanh(\beta\epsilon/2), \qquad -1 \leqslant \sigma_0 \leqslant 1. \tag{14}$$

We recall that σ_0 is known as the unsaturated inversion in laser theory (Haken 1970). In this paper we choose the ground state $|0\rangle_A\langle 0|$ defined by $\langle s_{1j}\rangle = \langle s_{2j}\rangle = 0$ and $\langle s_{3j}\rangle = -\frac{1}{2}$ as the initial atomic state $\rho_A(0)$. This choice corresponds to the limit $\beta \to \infty$ (zero temperature) or $\sigma_0 \to -1$. The case of finite temperature is treated in a subsequent paper.

4. Field dynamics

From the equation of motion (7) we now derive the time evolution of the field by eliminating the atomic variables. This reduction is achieved in terms of the partial trace with respect to the atomic system. Assuming zero detector temperature, we define the projector

$$\mathscr{P}_A = |0\rangle_A\langle 0| \otimes \mathrm{Tr}_A \tag{15}$$

leading to the reduced density operator

$$\rho_F(t) = \mathrm{Tr}_A\{\mathscr{P}_A \rho_D(t)\}. \tag{16}$$

We notice that $\mathscr{P}_A \mathscr{D}_A = 0$. Applying $\mathscr{P}_A$ to (7) we obtain the master equation

$$\dot{\rho}_F(t) = -\mathrm{i}[H_F, \rho_F(t)] + \lambda^2 V^{-1} \mathrm{Tr}_A \int_0^t \mathscr{P}_A \mathscr{H}_{AF} \mathscr{U}_{t'}^{\lambda} \mathscr{H}_{AF} \mathscr{P}_A \rho_D(t-t')\, \mathrm{d}t' \tag{17}$$

where

$$\mathscr{U}_t^{\lambda} \equiv \exp\{[\mathscr{H}_A + \mathscr{D}_A + \mathscr{H}_F + \lambda(\mathbb{1} - \mathscr{P}_A)\mathscr{H}_{AF}(\mathbb{1} - \mathscr{P}_A)]t\} \tag{18}$$

with $\mathscr{H}_{AF}$ denoting the Liouvillian corresponding to H_{AF} and with $\mathbb{1}$ standing for the unity Banach-space operator. The above equation is evaluated using $[\mathscr{H}_D, \mathscr{H}_{AB}] = 0$.

Dyson's formula leads to an expansion of (18) in powers of $\lambda N^{1/2}/\gamma V^{1/2}$ with the atomic damping constant $\gamma \equiv (\gamma_1 + \gamma_2)/2$. We assume that the atomic relaxation due to the bath is rapid compared with the relaxation due to the field, i.e. the corresponding characteristic times are supposed to obey

$$\gamma^{-1} \ll \lambda^{-1}(N/V)^{-1/2}. \tag{19}$$

Thus, we are entitled to use the Born approximation. This approximation is also compatible with our interest in solutions of (17) for times large compared with the atomic relaxation time, namely

$$t \gg \gamma^{-1}. \tag{20}$$

In the above limits, (17) leads to

$$\dot{\rho}_F(t) = -\mathrm{i}(\omega + \Delta)[a^\dagger a, \rho_F(t)] - \kappa[1 + (\epsilon - \omega)^2/\gamma^2]^{-1}([a^\dagger, a\rho_F(t)] + [\rho_F(t)a^\dagger, a]) \tag{21}$$

with the resonance absorption constant

$$\kappa \equiv \lambda^2 N/\gamma V \tag{22}$$

obeying $\kappa \ll \gamma$, and with the shift

$$\Delta \equiv \kappa\gamma^{-1}(\omega - \epsilon)[1 + (\omega - \epsilon)^2/\gamma^2]^{-1}. \tag{23}$$

Equation (21) can be transformed into a first-order partial differential equation for the quasi-probability $P_F(\alpha, t)$ defined by

$$\rho_F(t) \equiv \pi^{-1} \int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha| P_F(\alpha, t) \tag{24}$$

where $|\alpha\rangle$ denotes the coherent state. In the case of resonance $\omega = \epsilon$ this equation reads

$$\frac{\partial}{\partial t} P_F(\alpha, t) = \kappa\left(\frac{\partial}{\partial \alpha}[\alpha P_F(\alpha, t)] + \frac{\partial}{\partial \alpha^*}[\alpha^* P_F(\alpha, t)]\right). \tag{25}$$

Using (25) we can easily retrive $P_F(\alpha,0)$ and thus $\rho_F(0)$ from $P_F(\alpha, t)$. Equation (25) leads to exponential decay of the field amplitude α with a time constant κ.

From (25) we derive the equation of motion of the νth field moment, $\nu = 1, 2, 3, \ldots$, in the resonant case, namely

$$(\mathrm{d}/\mathrm{d}t)\langle a^{+\nu}a^{\nu}\rangle_t = -2\nu\kappa\langle a^{+\nu}a^{\nu}\rangle_t. \tag{26}$$

The non-resonant case is described by the same equation, but with κ replaced by $\kappa[1+(\omega-\epsilon)^2/\gamma^2]^{-1}$. Equation (26) explicitly shows the attenuation of the field due to the absorption process. We observe that the absorption constant κ does not only depend on the atom–field coupling constant λ, but also on the atom–bath relaxation constant γ. We recall that no restriction other than the natural time scales $\lambda^{-1}(N/V)^{-1/2} \gg \gamma^{-1} \gg \tau_j$ has been adopted here. We point out that condition (19) is crucial for preventing radiative decay.

5. Dynamics of atomic moments and relation to field moments

In the preceding section we presented the time evolution of the statistical state of the field due to the interaction with the detector atoms. Since our aim is the retrieval of field statistics from observed detector behaviour, we next have to find the relation between the statistical state of the field and that of the atomic system representing the detector. This is done in terms of relations between field moments $\langle a^{+\nu}a^{\nu}\rangle$ and atomic moments. The latter moments are defined as expectation values of the powers of the operator N_2 defined as

$$N_2 = S_3 + N/2 = \sum_{j=1}^{N} s_j^+ s_j^-. \tag{27}$$

The factorial moments $N_2^{(\nu)}$ of N_2 are evaluated with respect to the reduced density operator ρ_A of the atomic system.

We obtain ρ_A from the Dicke operator ρ_D obeying (7) by taking the partial trace with respect to field variables. Being interested in the deviations from the initial ground state we consider

$$\rho_A(t) - \rho_A(0) \equiv \mathrm{Tr}_F(\rho_D(t) - \rho_D(0)) = \mathrm{Tr}_F[(\mathbb{1} - \mathscr{P}_A)\rho_D(t)] \tag{28}$$

using the projector $\mathscr{P}_A$ defined by (15). We thus have the master equation

$$\begin{aligned}\rho_A(t) - \rho_A(0) &= \lambda V^{-1/2}\mathrm{Tr}_F \int_0^t \mathrm{d}t' \mathscr{U}^{\lambda}_{t-t'}(\mathbb{1} - \mathscr{P}_A)\mathscr{H}_{AF}\mathscr{P}_A\rho_D(t') \\ &= \sum_{m=1}^{\infty} (\lambda V^{-1/2})^m \mathrm{Tr}_F \int_0^t \mathrm{d}t_1 \ldots \int_0^t \mathrm{d}t_m \\ &\quad \times \mathscr{U}^0_{t-t_1}(\mathbb{1} - \mathscr{P}_A)\mathscr{H}_{AF} \ldots \mathscr{U}^0_{t_m}(\mathbb{1} - \mathscr{P}_A)\mathscr{H}_{AF}\mathscr{P}_A\rho_D(t_1),\end{aligned} \tag{29}$$

where

$$\mathscr{U}^0_t = \exp[(\mathscr{D}_A + \mathscr{H}_A + \mathscr{H}_F)t]. \tag{30}$$

Because of (16), equation (29) provides relationships between the atomic and field

expectation values. We calculate

$$\langle N_2\rangle_t = \mathrm{Tr}_A\left[\left(\sum_{j=1}^{N} s_j^+ s_j^-\right)(\rho_A(t)-\rho_A(0))\right], \tag{31}$$

evaluating (29) at resonance $\omega = \epsilon$ up to the second order in λ for $t \gg \gamma^{-1}$, and find the linear relation

$$\langle N_2\rangle_t = \zeta\langle a^\dagger a\rangle_t \tag{32}$$

with the proportionality constant

$$\zeta \equiv \lambda^2 N/\gamma^2 V = \kappa/\gamma \tag{33}$$

obeying $\zeta \ll 1$. This property of ζ justifies the second-order approximation in the evaluation of $\langle N_2\rangle_t$. Out of resonance, ζ has to be multiplied by $[1+(\omega-\epsilon)^2/\gamma^2]^{-1}$. Thus the photocurrent is found to be proportional to the field intensity at any time $t \gg \gamma^{-1}$.

The calculation of higher-order factorial moments is outlined in the appendix. The final result reads ($N \gg 1$)

$$\langle N_2^{(\nu)}\rangle_t \equiv \langle N_2(N_2-1)\ldots(N_2-\nu+1)\rangle_t = \zeta^\nu\langle a^{\dagger\nu}a^\nu\rangle_t. \tag{34}$$

This relation between atom and field statistics is the main result of the present paper and is basic to the calculation of the photocounting probability.

6. Photocounting probability

In this section we discuss the experimentally accessible counting rate. Combining (26) and (34) for the case $\nu = 1$ leads to the rate equation

$$-(\mathrm{d}/\mathrm{d}t)\langle a^\dagger a\rangle_t = 2\gamma\langle N_2\rangle_t, \tag{35}$$

which tells us that the time variation of the number of photons inside the cavity is equal to the number of excited atoms (photoelectrons) which leave the cavity in the time interval γ^{-1}. Since γ^{-1} is very small, $\gamma^{-1} \ll \kappa^{-1}$, we can furthermore deduce that the atoms follow the field adiabatically. We evaluate (35) by introducing the coherent state representation $P_F(\alpha, t)$ for the field. We define the number $N_A(t_0, t_0+T)$ of atoms excited in the time interval (t_0, t_0+T), i.e. the number of photocounts, namely

$$\begin{aligned} N_A(t_0, t_0+T) &\equiv 2\gamma\int_{t_0}^{t_0+T}\langle N_2\rangle_t\,\mathrm{d}t \\ &= 2\kappa\int \mathrm{d}^2\alpha\, P_F(\alpha, 0)\int_{t_0}^{t_0+T}|\alpha|^2\exp(-2\kappa t)\,\mathrm{d}t \\ &= \int \mathrm{d}^2\alpha P_F(\alpha, 0)|\alpha|_{t_0}^2[1-\exp(-2\kappa T)] \\ &= n_F(t_0)-n_F(t_0+T) \end{aligned} \tag{36}$$

with

$$|\alpha|_t^2 \equiv |\alpha|^2\exp(-2\kappa t) \tag{37}$$

and with

$$n_F(t) \equiv \int d^2\alpha P_F(\alpha, t)|\alpha|^2 = \int d^2\alpha P_F(\alpha, 0)|\alpha|_t^2 \tag{38}$$

denoting the average number of photons at time t. We thus find that the number of photocounts (excited atoms) in the time interval (t_0, t_0+T) is equal to the number of photons which are absorbed in the same time interval. This result is not unexpected since our model does not include mechanisms allowing for field losses.

We define the νth factorial moment of the photoelectron number in the time interval $t_0 < t < t_0+T$ by

$$N_A^{(\nu)}(t_0, t_0+T) \equiv (2\gamma)^\nu \int_{t_0}^{t_0+T} dt_1 \ldots \int_{t_0}^{t_0+T} dt_\nu \langle N_2(t_1)(N_2(t_2)-1)\ldots(N_2(t_\nu)-\nu+1)\rangle. \tag{39}$$

As we are dealing with a Markoffian stochastic process, all joint probability distributions are determined by the solution of equation (25). We thus obtain

$$N_A^{(\nu)}(t_0, t_0+T) = \int d^2\alpha P_F(\alpha, 0)|\alpha|_{t_0}^{2\nu}[1-\exp(-2\kappa T)]^\nu. \tag{40}$$

Of particular interest is the photocounting probability $p(n; t_0, t_0+T)$ which we define from $N_A^{(\nu)}$ by

$$N_A^{(\nu)}(t_0, t_0+T) = \sum_n n![(n-\nu)!]^{-1} p(n; t_0, t_0+T). \tag{41}$$

The photocounting probability is calculated by using the standard generating function

$$Q(x; t_0, t_0+T) \equiv \sum_n (1-x)^n p(n; t_0, t_0+T). \tag{42}$$

We finally obtain

$$p(n; t_0, t_0+T) = (n!)^{-1} \int d^2\alpha P_F(\alpha, 0)(\eta|\alpha|_{t_0}^2)^n \exp(-\eta|\alpha|_{t_0}) \tag{43}$$

with

$$\eta \equiv 1-\exp(-2\kappa T), \qquad \kappa \equiv \lambda^2 N/\gamma V. \tag{44}$$

We thus find that the probability $p(n; t_0, t_0+T)$ of observing n excited atoms in the time interval (t_0, t_0+T) equals the probability that n photons are absorbed in the same time interval. The photocounting probability (43) obeys a Pauli equation with time-dependent transition probabilities, namely

$$\frac{d}{dT}p(n; t_0, t_0+T) = 2\kappa n\eta^{-1}(1-\eta)p(n; t_0, t_0+T) - 2\kappa(n+1)\eta^{-1}(1-\eta)p(n+1; t_0, t_0+T), \tag{45}$$

the solution of which reads

$$p(n; t_0, t_0+T) = \sum_{m \geq n} \binom{m}{n} \eta^n (1-\eta)^{m-n} p_m(t_0) \tag{46}$$

where $p_m(t_0) \equiv p(m; t_0, t_0+\infty)$ denotes the probability of having m photons at the time t_0. In the limit $T \to 0$ we find $p(n; t_0, t_0+0) = \delta_{n_0}$. We recall the condition $t_0 \gg \gamma^{-1}$.

We observe that the form of equations (43) and (46) is identical to that of the previous results due to Mollow (1968) and Scully and Lamb (1969), which were recently reconsidered by Rousseau (1977) in terms of perturbation theory for independent detector atoms. However, the physical meaning of the attenuation constant γ in (43) and (46) is different. This constant depends not only on the atom–field interaction (coupling constant λ), but also on the atomic relaxation due to the coupling to the reservoirs (relaxation constant γ), whereas the corresponding parameter in the previous results allows only for atom–field interaction.

The constant γ represents a condensed description of the experimental set-up (responsible for e.g. the extraction of photoelectrons from the detector). Under the simplification that the measuring apparatus can be described by a single parameter, we have shown how this parameter affects the observed counting rate.

The initial statistical state $P_F(\alpha, t_0)$ of the field is retrieved from knowledge of the counting probability $p(n; t_0, t_0+T)$ by inversion of the integral transformation (43). This is achieved by modifying a previous procedure (Wolf and Mehta 1964) for the inversion of Mandel's formula. To this end, we define

$$F(x, t_0) \equiv \sum_{n=0}^{\infty} (\mathrm{i}x/\eta)^n p(n; t_0, t_0+T) \tag{47}$$

and obtain

$$\int_0^{2\pi} \mathrm{d}\varphi\, P_F(\alpha, t_0) = \pi^{-1} \exp(\eta|\alpha|^2) \int_{-\infty}^{+\infty} \mathrm{d}x\, \exp(-\mathrm{i}|\alpha|^2 x) F(x, t_0), \tag{48}$$

where φ denotes the phase angle of α. We notice that we can retrieve only the phase average of $P_F(\alpha, t_0)$ since only field intensity correlations are involved in (43).

The essential result of this paper is relation (34) between atomic and field moments. This relation is valid for times large compared with the atomic relaxation time γ^{-1}, where typically $\gamma^{-1} \ll 10^{-8}$ s. We can therefore reconstruct the initial field statistics from counting rates measured at practically any time. Our model settles the usual difficulties of detection theory such as radiative decay, saturation, and stimulating emission, and incorporates previous results on this subject. We finally point out that our detector scheme can be adapted to very general atomic systems without imposing unrealistic restrictions on the lifetime of the excited states.

Appendix 1

We are interested in the quantity

$$\langle N_2^{(\nu)} \rangle_t \equiv \sum_{\{i_s\}} \prod_{l=1}^{\nu} \Delta_{i_l, \{i_p\}} \langle s_{i_1}^+ \ldots s_{i_\nu}^+ s_{i_\nu}^- \ldots s_{i_1} \rangle_t, \tag{A.1}$$

where

$$\Delta_{i_l, \{i_p\}} \equiv (1-\delta_{i_l, i_1}) \ldots (1-\delta_{i_l, i_{l-1}})(1-\delta_{i_l, i_{l+1}}) \ldots (1-\delta_{i_l, i_\nu}). \tag{A.2}$$

From (29) we obtain

$$\langle N_2^{(\nu)}\rangle_t = \sum_{\{i_s\}}\sum_{\{j_r\}}\prod_{l=1}^{\nu}\Delta_{i_l,\{i_p\}}\mathrm{Tr}_{\mathrm{A,F}}\Big(s_{i_1}^+\ldots s_{i_\nu}^+ s_{i_\nu}^-\ldots s_{i_1}^-\int_0^\infty \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\ldots\int_0^{t_{2\nu-1}}\mathrm{d}t_{2\nu}$$
$$\times\mathscr{U}^0_{t_1-t_2}(\mathbb{1}-\mathscr{P}_\mathrm{A})\mathscr{H}_{\mathrm{AF},j_1}\ldots\mathscr{U}^0_{t_{2\nu}}(\mathbb{1}-\mathscr{P}_\mathrm{A})\mathscr{H}_{\mathrm{AF},j_{2\nu}}$$
$$\times|0\rangle_\mathrm{A}\langle 0|\otimes\rho_\mathrm{F}(t)\Big)+\mathrm{O}(\zeta^{\nu+1}), \tag{A.3}$$

where we define now

$$\mathscr{H}_\mathrm{AF}\equiv\sum_j\mathscr{H}_{\mathrm{AF},j}\equiv\lambda V^{-1/2}\sum_j[(as_j^+ + a^\dagger s_j^-), \qquad]. \tag{A.4}$$

Equation (A.3) is obtained under the following approximations.

(i) By $\zeta\gg 1$, terms of order higher than ν are eliminated. Moreover, all terms with $l<\nu$ vanish by application of $s_{i_l}^-$ to the vacuum.

(ii) Long time behaviour $t\gg\gamma^{-1}$ is considered.

Because of the definition $\mathscr{P}_\mathrm{A}\equiv|0\rangle_\mathrm{A}\langle 0|\otimes\mathrm{Tr}_\mathrm{A}$ also the terms of (A.3) containing $\mathscr{P}_\mathrm{A}$ vanish. This is shown by the same argument that has been used to eliminate the terms $l<\nu$ in the expansion (29). Thus we are left with the expression

$$\langle N_2^{(\nu)}\rangle_t = \sum_{\{i_s\}}\sum_{\{j_r\}}\prod_{l=1}^{\nu}\Delta_{i_l,\{i_p\}}\int_0^\infty\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\ldots\int_0^{t_{2\nu-1}}\mathrm{d}t_{2\nu}$$
$$\times\mathrm{Tr}_{\mathrm{A,F}}(s_{i_1}^+\ldots s_{i_\nu}^+ s_{i_\nu}^-\ldots s_{i_1}^-\mathscr{U}^0_{t_1-t_2}\mathscr{H}_{\mathrm{AF},j_1}\ldots\mathscr{U}^0_{t_{2\nu}}\mathscr{H}_{\mathrm{AF},j_{2\nu}}|0\rangle_\mathrm{A}\langle 0|\otimes\rho_\mathrm{F}(t)). \tag{A.5}$$

This result can be further simplified by some straightforward manipulations.

(i) From the structure of the commutator we infer that the only non-vanishing terms are of the form

$$\sum_{\{i_s\}}\sum_{\{j_r\}}\prod_{l,l'=1}^{\nu}\Delta_{i_l,\{i_p\}}\Delta_{j_{l'},\{j_{p'}\}}\mathrm{Tr}_{\mathrm{A,F}}(s_{i_1}^+\ldots s_{i_\nu}^+ s_{i_\nu}^-\ldots s_{i_1}^-\mathscr{U}^0_{t_1-t_2}\mathscr{H}_{\mathrm{AF},j_1}\ldots$$
$$\ldots\mathscr{U}^0_{t_{2\nu}}\mathscr{H}_{\mathrm{AF},j_{2\nu}}|0\rangle_\mathrm{A}\langle 0|\otimes\rho_\mathrm{F}). \tag{A.6}$$

(ii) Using the properties

$$\mathrm{Tr}(s_i^+ s_i^-\mathscr{U}_t^0 X)=\exp(-2\gamma t)\,\mathrm{Tr}(s_i^+ s_i^- X)\quad\text{(at zero temperature)},$$
$$\mathrm{Tr}(A\mathscr{H}_{\mathrm{AF},j}\mathscr{U}_t^0 X)=\exp(-\gamma t)\,\mathrm{Tr}(A\mathscr{H}_{\mathrm{AF},j}X),$$

and

$$\mathrm{Tr}(\mathscr{U}_t^0 X)=\mathrm{Tr}\,X,$$

We can bring all the $\mathscr{U}^0_{t_l-t_{l+1}}$ operators to the left of (A.3). As a result we obtain the factor

$$\int_0^\infty\mathrm{d}t_1\exp(-2\nu\gamma t_1)\int_0^{t_1}\mathrm{d}t_2\ldots\int_0^{t_{2\nu-1}}\mathrm{d}t_{2\nu}\exp(\gamma t_2)\ldots\exp(\gamma t_{2\nu})$$
$$=(2\nu!)^{-1}\gamma^{-2\nu}. \tag{A.7}$$

Collecting all these results we can write (A.5) as

$$\langle N_2^{(\nu)}\rangle_t = (2\nu!)^{-1}\gamma^{-2\nu}\sum_{i_1,\ldots,i_\nu}\sum_{k_1,\ldots,k_{2\nu}}\prod_{l=1}^{\nu}\Delta_{i_l\{i_s\}}\prod_{l=1}^{\nu}\Delta_{k_l\{k_s\}}$$
$$\times \mathrm{Tr}_{A,F}(s_{i_1}^{+}\ldots s_{i_\nu}^{+}s_{i_\nu}^{-}\ldots s_{i_1}^{-}\mathscr{H}_{AF,k_1}\ldots\mathscr{H}_{AF,k_{2\nu}}|0\rangle_A\langle 0|\rho_F(t)). \tag{A.8}$$

The 2ν-fold commutator in (A.8) is evaluated by using the expansion formula

$$(\mathscr{H}_{AF})^{2\nu}|0\rangle_A\langle 0|\rho_F = \sum_{m=0}^{2\nu}\binom{2\nu}{m}(-1)^m(H_{AF})^{2\nu-m}|0\rangle_A\langle 0|\rho_F(H_{AF})^m. \tag{A.9}$$

Inserting (A.9) into (A.8) we notice that the only term of (A.9) yielding a non-zero contribution is the one showing ν operators of the type $s_{i_l}^{+}$ operating on $|0\rangle_A\langle 0|$. Hence (A.8) leads to

$$\langle N_2^{(\nu)}\rangle_t = \gamma^{-2\nu}(\lambda V^{-1/2})^{2\nu}(\nu!)^{-2}\sum_{i_1,\ldots,i_\nu}\sum_{k_1,\ldots,k_{2\nu}}\prod_{l=1}^{\nu}\Delta_{i_l\{i_s\}}\prod_{l=1}^{\nu}\Delta_{k_l\{k_s\}}$$
$$\times \mathrm{Tr}_A(s_{k_{\nu+1}}^{-}\ldots s_{k_{2\nu}}^{-}s_{i_1}^{+}\ldots s_{i_\nu}^{+}s_{i_1}^{-}\ldots s_{i_\nu}^{-}s_{k_1}^{+}\ldots s_{k_\nu}^{+}|0\rangle_A\langle 0|)\langle a^{+\nu}a^{\nu}\rangle_t. \tag{A.10}$$

The above trace is easily evaluated and is found to include the factor $(\nu!)^2$ resulting from the number of combinations of $s_{i_l}^{+}$ and $s_{k_m}^{-}$ and $s_{i_l}^{-}$ and $s_{k_m}^{+}$. We finally carry out the sums in (A.10) and obtain

$$\langle N_2^{(\nu)}\rangle_t = (\lambda^2\gamma^{-2}V^{-1})^{\nu}N(N-1)\ldots(N-\nu+1)\langle a^{+\nu}a^{\nu}\rangle_t. \tag{A.11}$$

References

Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A **6** 2211–37

Arecchi F T and Degiorgio V 1972 *Laserhandbook* vol 1, eds F T Arecchi and E O Schulz-Dubois (Amsterdam: North-Holland) pp 191–264

Arnedo A and Rocca F 1974 *Z. Phys.* **269** 205–13

Bonifacio R, Schwendimann P and Haake F 1971 *Phys. Rev.* A **4** 302–13

Cummins H Z and Pike E R 1974 *Photon Correlation and Light Beating Spectroscopy, Nato Advanced Study Institute, Capri 1973* (New York: Plenum Press)

Dicke R H 1954 *Phys. Rev.* **93** 99–110

Favre C and Martin Ph 1968 *Helv. Phys. Acta* **41** 333–61

Glauber R 1965 *Quantum Optics and Electronics, Les Houches 1964* eds C de Witt, A Blandin and C Cohen-Tannoudji (New York: Gordon and Breach) pp 63–185

—— 1969 *Quantum Optics, Varenna Course XLIII* ed R Glauber (New York: Academic Press) pp 15–56

Haake F 1973 *Statistical Treatment of Open Systems by Generalized Master Equations, Springer Tracts in Modern Physics* vol 66 (Heidelberg: Springer)

Haken H 1970 *Laser Theory, Encyclopedia of Physics* ed S Flügge, vol 25/2c ed L Genzel (Heidelberg: Springer)

Helstrom C W 1972 *Progress in Optics* vol 10, ed E Wolf (Amsterdam: North-Holland) pp 289–369

Jaiswal A K and Agarwal G S 1969 *J Opt. Soc. Am.* **59** 1446–52

Kelley P L and Kleiner W H 1964 *Phys. Rev.* **136** A316–34

Lehmberg R H 1968 *Phys. Rev.* **167** 1152–8

Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley) chap. 6

Mandel L 1963 *Progress in Optics* vol 2, ed E Wolf (Amsterdam: North-Holland) pp 181–248

Mandel L and Meltzer D 1969 *Phys. Rev.* **188** 198–212

Mandel L, Sudarshan E C G and Wolf E 1964 *Proc. Phys. Soc.* **84** 435–44

Mandel L and Wolf E 1965 *Rev. Mod. Phys.* **37** 213–87

—— (eds) 1978 *Proc. 4th Rochester Conf. on Coherence and Quantum Optics* (New York: Plenum Press)

Mehta C L 1970 *Progress in Optics* vol 8, ed E Wolf (Amsterdam: North-Holland) pp 373–440

Mollow B R 1968 *Phys. Rev.* **168** 1896–919
Rocca F 1973 *Phys. Rev.* D **8** 4403–10
Rousseau M 1977 *J. Phys. A: Math Gen.* **10** 1043–7
Scharf G 1970 *Helv. Phys. Acta* **43** 806–28
Schwendimann P 1972 *Z. Phys.* **251** 244–53
Scully M O and Lamb W E 1969 *Phys. Rev.* **179** 368–74
Selloni A, Schwendimann P, Quattropani A and Baltes H P 1977 *Opt. Commun.* **22** 131–4
Watrasiewicz B M and Rudd M J 1975 *Laser Doppler Measurements* (London: Butterworth)
Wolf E and Mehta C L 1964 *Phys. Rev. Lett.* **13** 705–9
Zardecki A and Delisle C 1977 *Opt. Acta* **24** 241–51
Zwanzig R 1961 *Lectures in Theoretical Physics* vol III (New York: Interscience) pp 106–41